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Two-component one-dimensional gas with integrable open boundary conditions

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Abstract. The open nonlinear Schrödinger model with spin degrees of freedom is considered. We find two types of integrable open boundary conditions. We study the Bethe ansatz states of the model at the infinitely strong coupling limit. For some boundary conditions, the spins of the Bethe ansatz wavefunctions are all aligned up or down. For the other type of boundary conditions all the spin configurations are degenerate.

1. Introduction

One-dimensional impenetrable gases with spin degrees of freedom (and infinite U Hubbard model) have some interesting properties [1–4]. The model with open boundary conditions has an $SU(2)^N$ symmetry (where N is the number of particles) that is the direct product of the individual $SU(2)$ transformations for spin of each particle. For the model with periodic boundary conditions there is a smaller symmetry, namely a subgroup of $SU(2)^N$, being the $SU(2)^N$ transformation that does not change under cyclic permuations of the particles [3]. As the model has large symmetries, the ground state is degenerate in both boundary conditions.

Recently Izergin *et al* gave the Fredholm determinant representations for the correlation functions [5] of the impenetrable two-component Bose and Fermi gas system (and infinite U Hubbard model) with periodic boundary conditions [6–8], while Kojima has studied the correlation functions of the impenetrable Bose gases with Dirichlet and Neumann boundary conditions [9]. There has been increasing interest in integrable electron systems with open boundary conditions or with impurities in the condensed matter physics [11, 12, 15, 16].

For this paper, we studied the two-component Bose and Fermi gases with open boundary conditions. In the model there exist two types of integrable open boundary conditions, one corresponding to the existence of a boundary chemical potential while the other type is similar to a boundary condition with a boundary magnetic field.

We also studied the wavefunctions and spin configurations of the model at infinitely strong repulsive interactions.

The contents of this paper are as follows. In section 2, two-component one-dimensional gases with open boundary conditions at finite coupling are considered. Integrable open boundary conditions for the model are classified. The model at $c \to \infty$ is studied in section 3.

Details will be published elsewhere [19].

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2. Integrable boundary conditions for the two-component nonlinear Schrödinger model

The Hamiltonian for the two-component gas with open boundary conditions is given by

$$H = \int_0^L \{(\partial_x \psi^\dagger \partial \psi)(x) + c : (\psi^\dagger \psi)^2(x) :\} + c_0^+(\psi_+^\dagger \psi_+)(0) + c_0^-(\psi_-^\dagger \psi_-)(0)$$
$$+ c_L^+(\psi_+^\dagger \psi_+)(L) + c_L^-(\psi_-^\dagger \psi_-)(L) \tag{1}$$

where $\psi_\alpha(x)$ and $\psi_\alpha^\dagger(x)$ ($\alpha = \text{spin} \pm$) satisfy the canonical commutation relations for the boson or fermion, c is the coupling constant and c_0^α (c_L^α) for $\alpha = \pm$ are the boundary fields at the boundary $x = 0$ (respectively $x = L$). For the model with finite coupling constant c, the numbers of particles with spins up and down are conserved separately and then the eigenstates which contain $(N - M)$ particles of up spins and M particles of down spins can be written in the form

$$|\Psi_{N,M}\rangle = \int_0^L \mathrm{d}z_1 \ldots \int_0^L \mathrm{d}z_N \sum_{\alpha_1 \ldots \alpha_N = \pm} \chi_{N,M}^{\alpha_1 \ldots \alpha_N}(z_1, \ldots, z_N) \psi_{\alpha_1}^\dagger(z_1) \ldots \psi_{\alpha_N}^\dagger(z_N) |0\rangle \tag{2}$$

where $|0\rangle$ is the pseudovacuum.

To obtain the Bethe ansatz wavefunctions, it is covenient to consider the system of the particles on the circle $-L \leqslant z_j \leqslant L$ (extended space). There are boundary fields at $z_j = 0$ and $z_j = L$ and there are delta function interactions between both particle–particle and particle–mirror image (about the mirror crossing at 0 and L of the circle) of the particle. From the symmetry about $z_j \to -z_j$ the wavefunctions can be constructed to be invariant under this transformation. We can obtain the wavefunctions of the model with open boundary conditions by restricting coordinates z_j of the wavefunctions to be $0 \leqslant z_j \leqslant L$.

The first quantized Hamiltonian for the model on the extended space is given by

$$h = -\sum_{j=1}^N \partial_{z_j}^2 + c \sum_{i \neq j} [\delta(z_i - z_j) + \delta(z_i + z_j)] + c_0^+ \sum_{j=1}^N \delta(z_j) \delta_{\alpha_j,+}$$
$$+ c_0^- \sum_{j=1}^N \delta(z_j) \delta_{\alpha_j,-} + c_L^+ \sum_{j=1}^N \delta(z_j) \delta_{\alpha_j,+} + c_L^- \sum_{j=1}^N \delta(z_j) \delta_{\alpha_j,-}. \tag{3}$$

Bethe ansatz wavefunctions for the model with open boundary conditions have the following form:

$$\chi^{\alpha_1 \ldots \alpha_N}(z_1 \ldots z_N) = \mathcal{S} \sum_{a_1, \ldots, a_N = \pm} \mathrm{e}^{\mathrm{i} \sum_{j=1}^N a_j k_j z_j} \sum_{Q \in S_N} \xi_{\alpha_{Q(1)}, \ldots, \alpha_{Q(N)}}$$
$$\times (a_{Q(1)} k_{Q(1)}, \ldots, a_{Q(N)} k_{Q(N)} | Q) \theta(z_Q)$$
$$= \sum_{P \in S_N} \sum_{Q \in S_N} \sum_{a_1, \ldots, a_N = \pm} \epsilon^{|P|} \mathrm{e}^{\mathrm{i} \sum_j a_{P(j)} k_{P(j)} z_j} \xi_{\alpha_{Q(1)}, \ldots, \alpha_{Q(N)}}$$
$$\times (a_{PQ(1)} k_{PQ(1)}, \ldots, a_{PQ(N)} k_{PQ(N)} | PQ) \theta(z_Q) \tag{4}$$

where S_N is the symmetric group of order N, $\theta(z_Q)$ is 1 in the region z_Q (the region where $z_{Q(1)} < \cdots < z_{Q(N)}$) and 0 in the other regions, and $\mathcal{S}$ is the symmetrizer (antisymmetrizer) for the system of bosons (resp. fermions).

$$\mathcal{S} f_{\alpha_1 \ldots \alpha_N}(z_1 \ldots z_N) = \sum_{P \in S_N} \epsilon^{|P|} f_{\alpha_{P(1)} \ldots \alpha_{P(N)}}(z_{P(1)} \ldots z_{P(N)}) \tag{5}$$

where $\epsilon = +1$ for the Bose gas and $\epsilon = -1$ for the Fermi gas. In equation (4) $\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{Q(1)}k_{Q(1)},\ldots,a_{Q(N)}k_{Q(N)}|Q)$ are the spin-dependent amplitudes in the region z_Q and are the components of 2^N-dimensional vector $\xi(a_{Q(1)}k_{Q(1)},\ldots,a_{Q(N)}k_{Q(N)}|Q)$. Summation over a_j's are specific to the open boundary conditions. The wavefunctions for some coordinates z_j which have minus signs can be obtained by replacing z_j by $-z_j$ except for those in the θ function.

Let us define the S matrices and boundary S matrices as follows

$$\begin{aligned} S_{Q(i)Q(i+1)}&(a_{Q(i)}k_{Q(i)},a_{Q(i+1)}k_{Q(i+1)})\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(\ldots,a_{Q(i)}k_{Q(i)},a_{Q(i+1)}k_{Q(i+1)},\ldots|Q) \\ &= \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(i+1)},\alpha_{Q(i)},\ldots,\alpha_{Q(N)}}(\ldots,a_{Q(i+1)}k_{Q(i+1)},a_{Q(i)}k_{Q(i)},\ldots|(i,i+1)Q) \end{aligned} \tag{6}$$

$$\begin{aligned} K^0_{Q(1)}&(a_{Q(1)}k_{Q(1)})\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{Q(1)}k_{Q(1)},\ldots,a_{Q(N)}k_{Q(N)}|Q) \\ &= \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(-a_{Q(1)}k_{Q(1)},\ldots,a_{Q(N)}k_{Q(N)}|Q) \end{aligned} \tag{7}$$

$$\begin{aligned} K^L_{Q(N)}&(a_{Q(N)}k_{Q(N)})\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{Q(1)}k_{Q(1)},\ldots,a_{Q(N)}k_{Q(N)}|Q) \\ &= \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{Q(1)}k_{Q(1)},\ldots,-a_{Q(N)}k_{Q(N)}|Q) \end{aligned} \tag{8}$$

where subscripts denote the spin spaces of which the (boundary) S matrices act.

In other words S matrices determine the relation between the amplitudes of adjacent regions and boundary S matrices determine the relation between the amplitude of rapidity $k_{Q(1)}$ ($k_{Q(N)}$)and that of rapidity $-k_{Q(1)}$ ($-k_{Q(N)}$) for the region z_Q i.e. the relation between the amplitude of $z_{Q(1)} > 0$ ($z_{Q(N)} < L$) and that of $z_{Q(1)} < 0$ (resp. $z_{Q(N)} > L$). The fundamentals of the Bethe ansatz are the condition that for the integrable models, S matrices and boundary S matrices satisfy the Yang–Baxter equation and reflection equations, respectively.

The amplitude in the region z_Q can be obtained by successively applying the S matrices and boundary S matrices to the *reference amplitude* $\xi_{\alpha_1,\ldots,\alpha_N}(k_1,\ldots,k_N|I)$ where I denotes the region $z_1 < \cdots < z_N$. Moreover there are many paths from the reference amplitude to the amplitude in the region z_Q. To guarantee consistency for all the paths it is necessary not only that the S matrices (boundary S matrices) satisfy the Yang–Baxter equation (reflection equation) but also that the amplitudes (especially the reference amplitude) must satisfy the quantization conditions.

The quantization conditions (open boundary conditions) are found by considering the negation of $a_{P(j)}$

$$\begin{aligned} \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}&(a_{P(1)}k_{P(1)},\ldots,a_{P(j)}k_{P(j)},\ldots,a_{P(N)}k_{P(N)}|P) \\ &\Rightarrow \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{P(1)}k_{P(1)},\ldots,-a_{P(j)}k_{P(j)},\ldots,a_{P(N)}k_{P(N)}|P) \end{aligned} \tag{9}$$

along two different paths. In the one path particle $z_{Q(j)}$ is exchanged with particles $z_{Q(j-1)},\ldots,z_{Q(1)}$ by $S_{Q(j),Q(j-1)}\ldots S_{Q(j),Q(1)}$, reflected at $z_{Q(j)} = 0$ and again exchanged with particles $z_{Q(1)},\ldots,z_{Q(j-1)}$ by $S_{Q(j),Q(1)}\ldots S_{Q(j),Q(j-1)}$. In the other path particle $z_{Q(1)}$ is exchanged with particles $z_{j+1},\ldots,z_N$, is reflected at $z_{Q(j)} = L$ and finally exchanged with particles $z_{Q(N)},\ldots,z_{Q(j+1)}$.

These quantization conditions give the condition that the reference amplitude is the simultaneous eigenvector (as a 2^N-dimensional vector) of the operators

$$\begin{aligned} Z_j(\{a\},\{k\}) = {} & S_{j,j-1}(a_jk_j,a_{j-1}k_{j-1})\ldots S_{j,1}(a_jk_j,a_1k_1) \\ & \times K_0(a_jk_j)S_{1,j}(a_1k_1,-a_jk_j)\ldots S_{j-1,j}(a_{j-1}k_{j-1},-a_jk_j) \\ & \times S_{j+1,j}(a_{j+1}k_{j+1},-a_jk_j)\ldots S_{N,j}(a_Nk_N,-a_jk_j) \\ & \times K_L(a_jk_j)S_{j,N}(a_jk_j,a_Nk_N)\ldots S_{j,j+1}(a_jk_j,a_{j+1}k_{j+1}) \end{aligned} \tag{10}$$

for $j = 1,\ldots,N$. By solving the quantization conditions we get the (nested) Bethe ansatz equations.

The S matrices and the boundary S matrices are obtained by observing the boundary condition for the scatterings of particle–particle and for the crossing of particle to the point $0, L$. S matrices are obtained as

$$S_{ij}(a_i k_i, a_j k_j) = \frac{a_i k_i - a_j k_j + \mathrm{i}c\epsilon P_{ij}/2}{a_i k_i - a_j k_j - \mathrm{i}c/2}. \tag{11}$$

The boundary condition for the particle $z_{Q(1)}$ to cross the point 0 is given by

$$\begin{aligned}\partial_{z_{Q(1)}} \chi^{\alpha_1,\ldots\alpha_N}(z_1,\ldots,z_N)|_{z_Q(1)=-\epsilon}^{z_Q(1)=\epsilon} &= c_0^+ \delta_{\alpha_{Q(1)},+} \chi^{\alpha_1,\ldots\alpha_N}(\ldots, z_{Q(1)}=0,\ldots)\\ &\quad + c_0^- \delta_{\alpha_{Q(1)},-} \chi^{\alpha_1,\ldots\alpha_N}(\ldots, z_{Q(1)}=0,\ldots).\end{aligned} \tag{12}$$

From equations (12) and the boundary condition at $z_j = L$ we obtain the boundary S matrices as

$$\begin{aligned}K_i^0(a_i k_i) &= \begin{pmatrix} \frac{2\mathrm{i}a_i k_i - c_0^+}{2\mathrm{i}a_i k_i + c_0^+} & 0 \\ 0 & \frac{2\mathrm{i}a_i k_i - c_0^-}{2\mathrm{i}a_i k_i + c_0^-}\end{pmatrix}\\ K_i^L(a_i k_i) &= \exp 2\mathrm{i}a_i k_i L \begin{pmatrix} \frac{2\mathrm{i}a_i k_i - c_L^+}{2\mathrm{i}a_i k_i + c_L^+} & 0 \\ 0 & \frac{2\mathrm{i}a_i k_i - c_L^-}{2\mathrm{i}a_i k_i + c_L^-}\end{pmatrix}.\end{aligned} \tag{13}$$

For the general values of the boundary fields, boundary S matrices do not satisfy the reflection equation. Only special values of the boundary fields correspond to integrable boundary conditions [17]. There exist two types of integrable boundary conditions

$$\begin{array}{lll} \mathrm{I} & c_\gamma^+ = c_\gamma^- = c_\gamma & \\ \text{II-a} & c_\gamma^+ = \infty & \text{and } c_\gamma^- \text{ is arbitrary} \\ \text{II-b} & c_\gamma^- = \infty & \text{and } c_\gamma^+ \text{ is arbitrary} \end{array} \tag{14}$$

where $\gamma = 0, L$.

Type I boundary conditions correspond to the existence of boundary chemical potential and there are counterparts in the supersymmetric t–J model [14] and in the Hubbard model [13].

In type II boundary conditions boundary S matrices are somewhat similar to those of the Hubbard model with boundary magnetic fields [13].

Note that in the type II-a (II-b) boundary conditions if $c_\gamma^- = 0$ ($c_\gamma^+ = 0$) boundary S matrices are proportional to σ^z which are boundary S matrices for the XXZ model with an infinite strength boundary magnetic field [17].

3. Strong coupling limit

We shall now consider the *impenetrable* two-component particles ($c \to \infty$) with integrable open boundary conditions. At the $c \to \infty$ limit impenetrability yields

$$\begin{aligned}&\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(\ldots, a_{P(j)}k_{P(j)}, a_{P(j+1)}k_{P(j+1)},\ldots|P)\\ &\quad = -\epsilon \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(\ldots, a_{P(j+1)}k_{P(j+1)}, a_{P(j)}k_{P(j)},\ldots|(j, j+1)P)\end{aligned} \tag{15}$$

and therefore for all $P, Q, R \in S_N$

$$\begin{aligned}&\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{P(1)}k_{P(1)},\ldots, a_{P(N)}k_{P(N)}|P)\\ &\quad = (-\epsilon)^{|R|-|P|} \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{R(1)}k_{R(1)},\ldots, a_{R(N)}k_{R(N)}|R).\end{aligned} \tag{16}$$

Then the wavefunctions can be written in the form

$$\begin{aligned}\chi^{\alpha_1\ldots\alpha_N}(z_1\ldots z_N) &= \sum_{P\in S_N}\sum_{Q\in S_N}\sum_{a_1,\ldots,a_N=\pm}(-1)^{|P|}(-\epsilon)^{|Q|}\mathrm{e}^{\mathrm{i}\sum_{j=1}^{N}a_{P(j)}k_{P(j)}z_j}\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}\\ &\quad\times(a_1k_1,\ldots,a_Nk_N|I)\theta(z_Q)\\ &= \sum_{Q\in S_N}\sum_{a_1,\ldots,a_N=\pm}\det\{e^{a_ik_iz_j}\}\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_1k_1,\ldots,a_Nk_N|I)\theta(z_Q).\end{aligned} \tag{17}$$

Let us investigate the system with boundary conditions such that both left and right boundaries are of type II-b with $c^+ = 0$. With these boundary conditions boundary S matrices are of the form

$$\begin{aligned}&\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(-a_{P(1)}k_{P(1)},a_{P(2)}k_{P(2)},\ldots,a_{P(N)}k_{P(N)}|P)\\ &\qquad=\alpha_{Q(1)}\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{P(1)}k_{P(1)},a_{P(2)}k_{P(2)},\ldots,a_{P(N)}k_{P(N)}|P)\end{aligned} \tag{18}$$

$$\begin{aligned}&\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{P(1)}k_{P(1)},\ldots,-a_{P(N)}k_{P(N)}|P)\\ &\qquad=\alpha_{Q(N)}\mathrm{e}^{2\mathrm{i}k_{P(N)}L}\xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(a_{P(1)}k_{P(1)},\ldots,a_{P(N)}k_{P(N)}|P).\end{aligned} \tag{19}$$

Note that α_j's take values $+$ or $-$.

The quantization conditions are given as

$$\alpha_{Q(1)} = \exp(2\mathrm{i}k_{P(N)}L)\alpha_{Q(N)} \tag{20}$$

for any $P, Q \in S_N$. The solution to equations (20) for $N \geqslant 3$ is given by

$$\alpha_1 = \ldots = \alpha_N = \begin{cases} + \\ - \end{cases} \text{or} \qquad k_j = \pi n/L \tag{21}$$

where n is an integer. Then for this type of boundary condition simultaneous eigenvector of the operators $Z_j(\{a\},\{k\})$ are only trivial two states (highest or lowest states) $\xi_{+\ldots+}(a_1k_1,\ldots,a_Nk_N|I)$ i.e. all the particles have $+$ spins or $\xi_{-\ldots-}(a_1k_1,\ldots,a_Nk_N|I)$ i.e. all the particles are of spins $-$ for $N \geqslant 3$ (the case $N = 2$ is discussed in appendix B). With these boundary conditions symmetry of the spin configurations is broken by the boundary fields (the spin configurations of the two-component impenetrable models are discussed in appendix A) and therefore we cannot obtain the complete states from the Bethe ansatz and generators of the symmetry group. This situation is similar to the open XXX model (open Hubbard model) with boundary magnetic fields (resp. boundary fields) which do not have $SU(2)$ (resp. $SU(2)\times SU(2)/\mathbb{Z}_2$) symmetry.

Next we shall investigate the system with the boundary conditions such that both left and right boundaries are of type I with $c_\gamma^+ = c_\gamma^- = \infty$ or $c_\gamma^+ = c_\gamma^- = 0$. For the boundary conditions of type I, boundary S matrices do not depend on the spins and then the quantization conditions do not include spin degrees of freedom, i.e. $Z_j(\{a\},\{k\})$ are scalars. It follows that there is not any condition for the reference amplitude i.e. any 2^N-dimensional vectors serve as a reference amplitude to the Bethe ansatz wavefunctions.

The essential observation is that this seemingly peculiar result comes from the very large symmetry $SU(2)^N$ (see appendix A). That is, for the model with type I boundary conditions, all the spin configurations are degenerate.

For the Dirichlet boundary conditions ($c_\gamma^+ = c_\gamma^- = \infty$) boundary S matrices are obtained as

$$\begin{aligned}K^0(a_jk_j) &= -I\\ K^L(a_jk_j) &= -\exp(2\mathrm{i}a_jk_j)I\end{aligned} \tag{22}$$

where I is the identity matrix in the spin spaces. Quantization conditions for these boundary conditions are obtained as

$$\exp(2\mathrm{i}k_j) = 1 \tag{23}$$

for $j = 1, \ldots, N$.

The wavefunctions in the Dirichlet boundary conditions are given as

$$\chi^{\alpha_1,\ldots,\alpha_N}(z_1, \ldots, z_N|\{k\}) = (2\mathrm{i})^N \det\{\sin k_i z_j\} \sum_{Q\in S_N} (-\epsilon)^{|Q|} \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(k_1, \ldots, k_N|I)\theta(z_Q) \tag{24}$$

where the solutions of Bethe ansatz equations are expressed as

$$k_j = \frac{\pi}{L} n \qquad n \in \mathbb{Z}_{>0}. \tag{25}$$

For example the ground state configuration is given by

$$k_j = \frac{\pi}{L} j. \tag{26}$$

For the Neumann boundary conditions ($c_\gamma^+ = c_\gamma^- = 0$) boundary S matrices are given by

$$\begin{aligned} K^0(a_j k_j) &= I \\ K^L(a_j k_j) &= \exp(2\mathrm{i}a_j k_j) I \end{aligned} \tag{27}$$

and quantization conditions are the same as for those with Dirichlet boundary conditions. Then wavefunctions for the Neumann boundary conditions are given by

$$\chi^{\alpha_1,\ldots,\alpha_N}(z_1, \ldots, z_N|\{k\}) = (2i)^N \det\{\cos k_i z_j\} \sum_{Q\in S_N} (-\epsilon)^{|Q|} \xi_{\alpha_{Q(1)},\ldots,\alpha_{Q(N)}}(k_1, \ldots, k_N|I)\theta(z_Q) \tag{28}$$

where

$$k_j = \frac{\pi}{L} n \qquad n \in \mathbb{Z}_{\geqslant 0}. \tag{29}$$

The ground state configuration of the Neumann boundary conditions is

$$k_j = \frac{\pi}{L}(j-1). \tag{30}$$

Note that the ground state that we obtained has very high degeneracy (2^N states) but for the model with finite c the ground state is unique. Then among the degenerate states only one state corresponds to the ground state for the model with finite c.

4. Discussions and conclusions

The integrable boundary conditions for the spin nonlinear Schrödinger model are classified into two types. One type (type I) is related to the existence of boundary chemical potential, and the other (type II) is related to the existence of boundary magnetic field.

At the infinitely strong coupling limit, if at least one of the boundaries is of type II-a (II-b) with $c^- = 0$ ($c^+ = 0$) all the spins are parallel at least for the Bethe ansatz states. If both boundaries are of type I, there are not any conditions for the spin wavefunctions. In this case all the spin configurations are degenerate. This degeneracy is removed by an infinitesimal perturbation of the order $1/c$. In particular the ground state which we obtained has degeneracy from the spin configurations. But only one of these degenerate states corresponds to the ground state for the model with finite c.

To apply this model to the condensed matter physics, study of the ground state which is the $c \to \infty$ limit of the ground state of the model with finite coupling is important. Details of this point and studies of correlation functions in this ground state are being prepared for publication [19].

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Appendix A. Symmetry about spin configurations at $c \to \infty$

In this appendix we shall prove that at the infinitly strong coupling the two-component model with type I boundary conditions has a symmetry under the spin flippings of the individual spins of the particles independently.

For convenience, we shall consider the model on the lattice. For the model on the lattice (Hubbard-like model), the Hamiltonian of the impenetrable particles with spins can be written using the projection operator as

$$H = P\Big[-\sum_{j=1}^{L}\sum_{\alpha=\pm}(\psi^\dagger_{n+1,\alpha}\psi_{n,\alpha} + \psi^\dagger_{n,\alpha}\psi_{n+1,\alpha}) + c_0^+\psi^\dagger_{0,+}\psi_{0,+} + c_0^-\psi^\dagger_{0,-}\psi_{0,-} + c_L^+\psi^\dagger_{L,+}\psi_{L,+} + c_L^-\psi^\dagger_{L,-}\psi_{L,-}\Big]P \tag{31}$$

where P is the projector onto the states where double occupations are forbidden.

Let us introduce the spin operators for the jth particle $S_{j,\gamma}$ such that $S_{j,\gamma} = \sigma_n^\gamma$ if jth particle lies on the nth site where $\gamma = x, y, z,\ j = 1, \ldots, N,\ n = 1, \ldots, L$ and σ_n^γ act as Pauli matrices for the spin of the nth particle. It is known that kinetic terms commute with $S_{j,\gamma}$ [3],

$$\Big[S_{j,\gamma}, P\sum_{\alpha}(\psi^\dagger_{n+1,\alpha}\psi_{n,\alpha} + \psi^\dagger_{n,\alpha}\psi_{n+1,\alpha})P\Big] = 0. \tag{32}$$

Also $S_{j,\gamma}$'s commute with boundary chemical potential but they do not commute with boundary magnetic field:

$$\begin{aligned} &[S_{j,\gamma}, P(\psi^\dagger_{a,+}\psi_{a,+} + \psi^\dagger_{a,-}\psi_{a,-})P] = 0 \\ &[S_{j,\gamma}, P\psi^\dagger_{a,\alpha}\psi_{a,\alpha}P] \neq 0 \end{aligned} \tag{33}$$

where $\alpha = \pm$ and $a = 0, L$. Thus for type I boundary conditions, the model has symmety that flips the spins of the individual particles independently. And all the spin configurations are degenerate. This degeneracy agrees with intuition. This symmetry is a much larger symmetry compared with the spin $SU(2)$ symmetry for the model with finite strength coupling.

The boundary conditions of type II break the above symmetry.

Note that for the model with periodic boundary conditions, the Hamiltonian commutes with a subgroup of $SU(2)^N$. That is the spin configurations are degenerate modulo-cyclic permutations.

Appendix B. Boundary conditions of type II

Here we consider the systems with boundary conditions of type II-b at $c^+ = 0$ for both boundaries with $N = 2$ explicitly. The wavefunctions for the system with two particles are given by

$$\begin{aligned}\chi^{\alpha_1\alpha_2}(z_1, z_2) = &\{(\mathrm{e}^{\mathrm{i}k_1z_1} + \alpha_1\mathrm{e}^{-\mathrm{i}k_1z_1})(\mathrm{e}^{\mathrm{i}k_2z_2} + \alpha_2\mathrm{e}^{2\mathrm{i}k_2L-\mathrm{i}k_2z_2})\\ &-(\mathrm{e}^{\mathrm{i}k_1z_2} + \alpha_1\mathrm{e}^{-\mathrm{i}k_1z_2})(\mathrm{e}^{\mathrm{i}k_2z_1} + \alpha_2\mathrm{e}^{2\mathrm{i}k_2L-\mathrm{i}k_2z_1})\}\xi_{\alpha_1\alpha_2}(k_1, k_2)\theta(z_1 < z_2)\\ &-\epsilon\{(\mathrm{e}^{\mathrm{i}k_1z_1} + \alpha_2\mathrm{e}^{-\mathrm{i}k_1z_1})(\mathrm{e}^{\mathrm{i}k_2z_2} + \alpha_1\mathrm{e}^{2\mathrm{i}k_2L-\mathrm{i}k_2z_2})\\ &-(\mathrm{e}^{\mathrm{i}k_1z_2} + \alpha_2\mathrm{e}^{-\mathrm{i}k_1z_2})(\mathrm{e}^{\mathrm{i}k_2z_1} + \alpha_1\mathrm{e}^{2\mathrm{i}k_2L-\mathrm{i}k_2z_1})\}\xi_{\alpha_2\alpha_1}(k_1, k_2)\theta(z_2 < z_1).\end{aligned} \tag{34}$$

Then for the system with $\alpha_1 = -$, $\alpha_2 = +$

$$\begin{aligned}\chi^{-+}(z_1, z_2) = &-4\{\sin(k_1z_1)\sin(k_2z_2) - \sin(k_1z_2)\sin(k_2z_1)\}\xi_{-+}(k_1, k_2)\theta(z_1 < z_2)\}\\ &-4\epsilon\{\cos(k_1z_1)\cos(k_2z_2) - \cos(k_1z_2)\cos(k_2z_1)\}\xi_{+-}(k_1, k_2)\theta(z_2 < z_1).\end{aligned} \tag{35}$$

If the rapidities k_j's are given as $\mathrm{e}^{2\mathrm{i}k_jL} = -1$ for $j = 1, 2$, then the above wavefunctions satisfy the boundary conditions.

But for the three-particle case we we cannot consistently construct the wavefunctions of these forms and satisfy the boundary conditions. These observations confirm the fact that for the boundary conditions of type II at both ends (or at one end) and for $N \geqslant 3$, we cannot obtain the nonzero wavefunction when + spins and − spins are mixed (at least as the Bethe ansatz states).

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